ON H-CLOSED PARATOPOLOGICAL GROUPS

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A Hausdorff paratopological group is it H-closed if it is closed in every Hausdorff paratopological group containing it as a paratopological subgroup. Obtained a criterion when abelian topological group is H-closed and for some classes of abelian paratopological groups are obtained simple criteria of H-closedness.

Key words: *paratopological group, minimal topological group, absolutely closed topological group.*

All topological spaces considered in this paper are Hausdorff, if the opposite is not stated. We shall use the following notations. Let A be a subset of a group and n be an integer. Put $A^n = \{a_1 a_2 \cdots a_n : a_i \in A\}$ and $nA = \{a^n : a \in A\}$. For a group topology τ the closure of set A we define as $\overline{A}^\tau$ and the base of the unit as $\mathcal{B}_\tau$.

A topological space X of a class C of topological spaces is *C-closed* provided X is closed in any space Y of the class C containing X as a subspace. It is well known that when C is the class of Tychonoff spaces than C-closedness coincides with compactness. For the class of Hausdorff spaces the following conditions for a space X are equivalent [1, 3.12.5]

(1) The space X is H-closed.

(2) If $\mathcal{V}$ is a centered family of open subsets of X then $\bigcap\{\overline{V} : V \in \mathcal{V}\} \neq \varnothing$.

(3) Every ultrafilter in the family of all open subsets of X is convergent.

(4) Every cover $\mathcal{U}$ of the space X contains a finite subfamily $\mathcal{V}$ such that $\bigcup\{\overline{V} : V \in \mathcal{V}\} = X$.

The group G with topology τ is called a *paratopological group* if the multiplication on the group G is continuous. If the inversion on the group G is continuous then (G, τ) is a *topological group*. A group (G, τ) is paratopological if and only if the following conditions (known as Pontrjagin conditions) are satisfied for base $\mathcal{B}$ at unit e of G [4,5].

1. $\bigcap\{UU^{-1} : U \in \mathcal{B}\} = \{e\}$.
2. $(\forall U, V \in \mathcal{B})(\exists W \in \mathcal{B}) : W \subset U \cap V$.
3. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^2 \subset U$.
4. $(\forall U \in \mathcal{B})(\forall u \in U)(\exists V \in \mathcal{B}) : uV \subset U$.
5. $(\forall U \in \mathcal{B})(\forall g \in G)(\exists V \in \mathcal{B}) : g^{-1}Vg \subset U$.

The paratopological group G is a topological group if and only if

6. $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B}) : V^{-1} \subset U$.

A topological group is *absolutely closed* if it is closed in every Hausdorff topological group containing it as a topological subgroup. A topological group G is H-closed if and only if it is *Rajkov-complete*, that is complete with respect to the upper uniformity which is defined as the least upper bound $\mathcal{L} \vee \mathcal{R}$ of the left and the right uniformities on G. Recall that the sets $\{(x, y) : x^{-1}y \in U\}$, where U runs over a base at unit of G, constitute a base of entourages for the left uniformity $\mathcal{L}$ on G. In the case of the

right uniformity $\mathcal{R}$, the condition $x^{-1}y \in U$ is replaced by $yx^{-1} \in U$. The *Rajkov completion* $\hat{G}$ of a topological group G is the completion of G with respect to the upper uniformity $\mathcal{L} \vee \mathcal{R}$. For every topological group G the space $\hat{G}$ has a natural structure of a topological group. The group $\hat{G}$ can be defined as a unique (up to an isomorphism) Rajkov complete group containing G as a dense subgroup.

A paratopological group is *H-closed* if it is closed in every Hausdorff paratopological group containing it as a subgroup. In the present section we shall consider H-closed paratopological groups.

Question Let G be a regular paratopological group which is closed in every regular paratopological group containing it as a subgroup. Is G H-closed?

1. Lemma. *Let (G,τ) be a paratopological group. If there exists a paratopology σ on the group $G \times \mathbb{Z}$ such that $\sigma|G \subset \tau$ and $e \in \overline{(G,1)}^{\sigma}$ then (G,τ) is not H-closed.*

Proof. We shall build the paratopology ρ on the group $G\times\mathbb{Z}$ such that $\rho|G = \tau$ and $\overline{G}^{\rho} \neq G$. Determine the base of unit $\mathcal{B}_\rho$ as follows. Let $S = \{(x,n) : x \in G, n > 0\}$. For every neighborhoods $U_1 \in \tau$, $U_2 \in \sigma$ such that $U_1 \subset U_2$ put $(U_1,U_2) = U_1 \cup (U_2 \cap S)$. Put $\mathcal{B}_\rho = \{(U_1,U_2) : U_1 \in \mathcal{B}_\tau, U_2 \in \mathcal{B}_\sigma\}$. Verify that $\mathcal{B}_\rho$ satisfies the Pontrjagin conditions.

1. It is satisfied since $(U_1,U_2) \subset U_2$.

2. It is satisfied since $(U_1 \cap V_1, U_2 \cap V_2) \subset (U_1,U_2) \cap (V_1,V_2)$.

3. Select $V_2 \in \mathcal{B}_\sigma$ and $V_1 \in \mathcal{B}_\tau$ such that $V_2^2 \subset U_2$, $V_1^2 \subset U_1$ and $V_1 \subset V_2$. Let $y_1, y_2 \in (V_1,V_2)$. The following cases are possible

A. $y_1, y_2 \in V_1$. Then $y_1y_2 \in V_1^2 \in (U_1,U_2)$.

B. $y_1 \in V_1, y_2 \in V_2 \cap S$. Then $y_1y_2 \in V_2^2 \in U_2$. Since $y_1 \in G$ and $y_2 \in S$ then $y_1y_2 \in S$ and hence $y_1y_2 \in U_2 \cap S$.

C. $y_1 \in V_2 \cap S, y_2 \in V_1$ is similar to the case B.

D. $y_1, y_2 \in V_2 \cap S$. Since S is a semigroup then $y_1y_2 \in U_2 \cap S$.

4. Let $y \in (U_1,U_2)$. There exist $V_2 \in \mathcal{B}_\sigma$ and $V_1 \in \mathcal{B}_\tau$ such that $yV_2 \subset U_2$ and $V_1 \subset V_2$. The following cases are possible

A. $y \in U_1$. We may suppose that $yV_1 \subset U_1$. Since $y \in G$ then $y(V_2 \cap S) \subset U_2 \cap S$.

B. $y \in U_2 \cap S$. Since $V_1 \subset G$ then $yV_1 \in U_2 \cap S$. Since S is a semigroup and $y \in S$ then $y(V_2 \cap S) \subset U_2 \cap S$. Therefore $y(V_1,V_2) \subset (U_1,U_2)$.

5. Let $(g,n) \in G \times \mathbb{Z}$. There exist $V_2 \in \mathcal{B}_\sigma$ and $V_1 \in \mathcal{B}_\tau$ such that $V_1 \subset V_2$, $g^{-1}V_1g \subset U_1$ and $g^{-1}V_2g \subset U_2$. Then $(g,n)^{-1}(V_1,V_2)(g,n) = g^{-1}(V_1,V_2)g = g^{-1}(V_1 \cup (V_2 \cap S)g \subset U_1 \cup (U_2 \cap S) = (U_1,U_2)$.

Therefore (H,ρ) is a paratopological group. Since $(U_1,U_2) \cap G = U_1$ then $\rho|G = \tau$.

Since $e \in \overline{(G,1)}^{\sigma}$ then for every $U_2 \in \mathcal{B}_\sigma$ there exists $g \in G$ such that $(g,1) \in U_2$. Then $g \in (e,-1)(U_2 \cap S)$ and therefore $(e,-1) \in \overline{G}^{\rho}$. $\square$

A group topology τ_1 on the group G is called complementable if there exist a nondiscrete group topology τ_2 on G and neighborhoods $U_i \in \tau_i$ such that $U_1 \cap U_2 = \{e\}$. In this case we say that τ_2 is a *complement* to τ_1. Proposition 1.4 from [1] implies that in this case a topology $\tau_1 \wedge \tau_2$ is Hausdorff.

A Banach measure is a real function μ defined on the family of all subsets of a group G which satisfies the following conditions:

(a) $\mu(G) = 1$.

(b) if $A, B \subset G$ and $A \cap B = \varnothing$ then $\mu(A \cup B) = \mu(A) + \mu(B)$.

(c) $\mu(gA) = \mu(A)$ for every element $g \in G$ and for every subset $A \subset G$.

2. Lemma. [3, p.37]. *Let G be an abelian group and let μ be a Banach measure on G. Let τ be a group topology on G. Suppose that the set nG is U-unbounded for some natural number n and for some neighborhood U of zero in (G,τ). Then $\mu(\{x\in G : nx\in gW\})=0$ for every element $g\in G$ and for every neighborhood W of zero satisfying $WW^{-1}\subset U$.*

Let U be a neighborhood of zero in a topological group (G,τ). We say that a subset $A\subset G$ is U-unbounded if $A\not\subset KU$ for every finite subset $K\subset G$.

Given any elements $a_0,a_1,\dots,a_n$ of an abelian group G put

$$Y(a_0,a_1,\dots,a_n)=\{a_0^{x_0}a_1^{x_1}\cdots a_n^{x_n} : 0\leqslant x_i\leqslant i+1, i\leqslant n, \sum x_i^2>0\},$$

$$X(a_0,a_1,\dots,a_n)=\{a_0^{x_0}a_1^{x_1}\cdots a_n^{x_n} : -(i+1)\leqslant x_i\leqslant i+1, i\leqslant n\}.$$

Then $X(a_0,a_1,\dots,a_n)=Y(a_0,a_1,\dots,a_n)Y(a_0,a_1,\dots,a_n)^{-1}$.

3. Lemma. *Let (G,τ) be an abelian paratopological group of the infinite exponent. If there exists a neighborhood $U\in\mathcal{B}_\tau$ such that a group nG is UU^{-1}-unbounded for every natural number n then the paratopological group (G,τ) is not H-closed.*

Proof. Define a seminorm $|\cdot|$ on the group G such that for all $x,y\in G$ holds $|xy|\leqslant|x|+|y|$. Suppose that there exists a non periodic element $x_0\in G$. Determine a map $\phi_0:\langle x_0\rangle\to\mathbb{Z}$ putting $\phi_0(x_0^n)=n$. Since $\mathbb{Q}$ is a divisible group then the map ϕ_0 can be extended to a homomorphism $\phi:G\to\mathbb{Q}$. Put $|x|=|\phi(x)|$ for every element $x\in G$. If G is periodic then put $|e|=0$ and $|x|=[\ln ord(x)]+1$, where $ord(x)$ denotes the order of the element x.

Fix a neighborhood $V\in\mathcal{B}_\tau$ such that $V^2\subset U$ and put $W=VV^{-1}$. We shall construct a sequence $\{a_n\}$ such that

(a) $|a_n|>n$.
(b) $W\cap X(a_0,a_1,\dots,a_n)=\{e\}$.
(c) $Y(a_0,a_1,\dots,a_n)\not\ni e$.
(d) if $-n\leqslant k\leqslant n, k\neq 0$ then $a_n^k\notin 2X(a_0,a_1,\dots,a_{n-1})$.

Take any element $a_0\notin W$. Suppose that the elements $a_0,\dots,a_n$ have been chosen satisfying conditions (a) and (b). Put

$$B_n=\{x\in G : (\forall g\in X(a_0,a_1,\dots,a_{n-1}))(\forall k\in\mathbb{Z}\backslash\{0\} : -e^{n+1}\leqslant k\leqslant e^{n+1}) : kx\notin gW\}.$$

If the group G is periodic then $|x|>n$ for every element $x\in B_n$. Lemma 2 implies that $\mu(B_n)=1$. If the group G is not periodic then the construction of the seminorm $|\cdot|$ implies that $\mu(\{x\in G : |x|\leqslant n\})=\mu(\phi^{-1}[-n;n])=0$. In both cases there exists an element $a_n\in B_n$ such that $|a_n|>n$. Then $W\cap X(a_0,a_1,\dots,a_n)=\varnothing$. Considering a subsequence and applying condition (a) we can satisfy conditions (c) and (d) also.

Define a base $\mathcal{B}_{\tau\{a_n\}}$ at the unit of group topology $\tau\{a_n\}$ on the group $G\times\mathbb{Z}$ as follows. Put $A_n^+=\{(e,0)\}\cap\{(a_k,1) : k\geqslant n\}$. For every increasing sequence $\{n_k\}$ put $A[n_k]=\bigcup_{l\in\mathbb{N}}A_{n_1}^+\cdots A_{n_l}^+$. Put $\mathcal{B}_{\tau\{a_n\}}=\{A[n_k]\}$. We claim that $(G\times Z,\tau\{a_n\})$ is a zero dimensional paratopological group.

Put $F=\bigcup_{n\in\omega}X(a_0,a_1,\dots,a_n)$. Let $A[n_k]\in\mathcal{B}_{\tau\{a_n\}},(x,n_x)\notin A[n_k]$. If $x\notin F$ then $(x,n_x)A[n]\cap A[n_k]=\varnothing$. Let $x\in X(a_0,a_1,\dots,a_m)$. Put $m_k=m+k$. Suppose that

$(x,n_x)A[m_k]\cap A[n_k]\neq\varnothing$. Select the minimal k such that $(x,n_x)(A^+_{m_1}\cdots A^+_{m_k})\cap A[n_k]\neq\varnothing$. Let

$$(*)\qquad (x,n_x)(a_{l_1},1)\cdots(a_{l_k},1)=(a_{l'_1},1)\cdots(a_{l'_{k'}},1)$$

and for all i,i' holds $m_i\leqslant l_i\leqslant l_{i+1}$, $n'_i\leqslant l'_{i'}\leqslant l'_{i'+1}$. Remark that a member a_q occurs in each part of the equality $(**)$ no more than q times. If $l_k>l'_{k'}$ then if we move all members which are not equal to $(a_{l_k},1)$ from the left side of the equality (*) to the right one, we obtain contradiction to condition (d). The case $l_k<l'_{k'}$ is considered similarly. Therefore $l_k=l'_{k'}$, a contradiction to that k is the minimal number such that the equality (*) holds. It is showed similarly that if $x\neq e$ and $m_k=m+k+1$ then $(x,n_x)\notin A[m_k]$. If $x=e$ and $n_x\neq 0$ then condition (c) implies that $A[n]\not\ni(x,n_x)$. Hence Pontrjagin condition 1 for $\mathcal{B}_{\tau\{a_n\}}$ is satisfied. Since $A[n_{2k}]^2\subset A[n_k]$, Pontrjagin condition 3 is satisfied. All other Pontrjagin conditions are obvious.

Condition (b) implies that $A[n]A[n]^{-1}\cap VV^{-1}=\{(e,0)\}$. Therefore the topology $\tau\{a_n\}_g$ is a complement to the topology $(\tau\times\{0\})_g$, where $\tau\times\{0\}$ is the product topology on the group $(G,\tau)\times\mathbb{Z}$. Therefore the topology $\sigma=\tau\{a_n\}(\tau\times\{0\})$ is Hausdorff. Since $(e,0)\in\overline{(G,1)}^{\tau\{a_n\}}\subset\overline{(G,1)}^{\sigma}$ then (G,τ) is not H-closed. $\square$

We shall need the following lemma.

4. Lemma. *Let G be a paratopological group and H be a normal subgroup of the group G. If H and G/H are topological groups then G is a topological group.*

Proof. Let U be an arbitrary neighborhood of the unit. There exist neighborhoods V,W of the unit such that $V\subset U$, $(V^{-1})^2\cap H\subset U$ and $W\subset V$, $W^{-1}\subset VH$. If $x\in W^{-1}$ then there exist elements $v\in V, h\in H$ such that $x=vh$. Then $h=v^{-1}x\in V^{-1}W^{-1}\cap H\subset U$. Therefore $x\in VU\subset U^2$. Hence G is a topological group. $\square$

5. Theorem. *An abelian topological group (G,τ) is H-closed if and only if (G,τ) is Rajkov complete and for every group topology $\sigma\subset\tau$ on G the quotient group $\hat{G}/G$ is periodic, where $\hat{G}$ is the Rajkov completion of the group (G,σ).*

Proof. Suppose that there exists a group topology $\sigma\subset\tau$ on G such that the quotient group $\hat{G}/G$ is not periodic, where $\hat{G}$ is the Rajkov completion of the group (G,σ). Select a non periodic element $x\in\hat{G}$ such that $\langle x\rangle\cap G=\{e\}$. Then $G\times\langle x\rangle$ is naturally isomorphic to a group $G\times\mathbb{Z}$ and Lemma 1 implies that the group (G,τ) is not H-closed.

Let a paratopological group (H,τ') contains (G,τ) as non closed subgroup. Since G is abelian then $\overline{G}$ is an abelian semigroup. Choose an arbitrary element $x\in\overline{G}\backslash G$. Then a group hull $F=\langle G,x\rangle$ with a topology $\tau'|F$ is an abelian paratopological group. Then the group G is dense in $(F,{\tau'}_g)$. Since the Rajkov completion $\hat{F}$ of the topological group $(F,\tau'|F_g)$ is periodic then there exists a natural number n such that $x^n\in G$. Therefore $F^n\subset G$. Lemma 4 implies that F is a topological group and therefore G is closed in $(F,{\tau'}_g)$, a contradiction. $\square$

6. Corollary. *A Rajkov completion of a isomorphic condensation of H-closed abelian topological group is H-closed.*

7. Proposition. *Let G be a Rajkov complete topological group, H be H-closed paratopological subgroup of the group G. If a group G/H has finite exponent then G is an H-closed paratopological group.*

Proof. Select a number n such that $g^n \in H$ for every element $g \in G$. Let $F \supset G$ be a paratopological group. Since H is closed in F then for every element $g \in \overline{G}$ holds $g^n \in H$. Denote continuous maps $\phi : \overline{G} \to \overline{G}$ as $\phi(g) = g^{n-1}$ and $\psi : \overline{G} \to H$ as $\psi(g) = (g^n)^{-1}$. Then for every element $g \in \overline{G}$ holds $g^{-1} = \phi(g)\psi(g)$ and therefore the inversion on the group $\overline{G}$ is continuous. Since $\overline{G}$ is a topological group and G is Rajkov complete then $\overline{G} = G$. □

8. Proposition. *Let G be a paratopological group and K be a compact normal subgroup of the group G. If a group G/K is H-closed then the group G is H-closed.*

Proof. Suppose that there exists a paratopological group F containing the group G such that $\overline{G} \neq G$. Since K is compact then F/K is a Hausdorff paratopological group by Proposition 1.13 from [4]. Let $\pi : F \to F/K$ be the standard map. Then $\overline{G/K} \supset \pi(\overline{\pi^{-1}(G/K)}) \supset \pi(\overline{G}) \neq \pi(G) = G/K$. This implies that the group G/K is not H-closed, a contradiction. □

Let G be a topological group, N be a closed normal subgroup of the group G. Then if N and G/N are Rajkov complete so is the group G [5]. This suggests the following

9. Question. Let G be a paratopological group, N be a closed normal subgroup of the group G and the groups N and G/N are H-closed. Is the group G H-closed?

Let (G, τ) be a paratopological group. Then there exists the finest group topology τ_g coarser than τ (see [2]), which is called *the group reflection* of the topology τ.

10. Proposition. *Let (G, τ) be an abelian paratopological group. If (G, τ_g) is H-closed then (G, τ) is H-closed. If (G, τ) is H-closed and (G, τ_g) is Rajkov complete then (G, τ_g) is H-closed.*

Proof. Suppose that the group (G, τ_g) is H-closed and (G, τ) is not. Let a paratopological $(H, \hat{\tau})$ contains (G, τ) as non closed subgroup. Without loss of generality we may suppose that there exists an element $x \in H\backslash G$ such that $H = \langle G, x\rangle$ and the group H is abelian. Let $\hat{\tau}_g$ be the group reflection of the topology $\hat{\tau}$. Since $\hat{\tau}_g|G \subset \tau_g$ then Theorem 5 implies that the group H/G is periodic. Without loss of generality we may suppose that $x^p \in G$ for some prime p.

Denote the family of neighborhoods at unit in the topology τ as $\mathcal{B}_{\hat{\tau}}$. Let $U \in \mathcal{B}_{\hat{\tau}}$. If $U \cap xG = \varnothing$ then there exists a neighborhood V of unit such that $V^p \subset U$ and thus $V \subset G$ and G is open in $(H, \hat{\tau})$. Therefore a set $\mathcal{F} = \{x^{-1}(xG \cap U) : U \in \mathcal{B}_\tau\}$ is a filter. Let $U \in \mathcal{B}_{\hat{\tau}}$. There exists $V \in \mathcal{B}_{\hat{\tau}}$ such that $V^p \subset U$. Then $(xG \cap V)^p \subset U$. Let $xg \in (xG \cap V)$. Then $x^{-1}(xG \cap V) \subset x^{-1}((xg)^{1-p}(xG \cap V)^p) \cap G \subset x^{-p}g^{1-p}(U \cap G)$ and hence $\mathcal{F}$ is a Cauchy filter in the group (G, τ_g). Let $h \in G$ be a limit of the filter $\mathcal{F}$ on the group (G, τ_g). But then for every neighborhood of the unit U in the topology $\hat{\tau}_g$ holds $U \cap xhU \supset U \cap xh(U \cap G) \neq \varnothing$ and therefore $(H, \hat{\tau}_g)$ is not Hausdorff, a contradiction.

Let (G, τ_g) is Rajkov complete and (G, τ_g) is not H-closed. Then Theorem 5 implies that there exists a group topology $\sigma \subset \tau$ on G such that the quotient group $\hat{G}/G$ of the Rajkov completion $\hat{G}$ of the group (G, σ) is not periodic. Then Lemma 1 implies that a group (G, τ) is not H-closed. □

11. Lemma. *Let topological group (H,σ_H) be a closed subgroup of an abelian topological group (G,τ) and $\sigma_H\subset\tau|H$. Then there exists a group topology $\sigma\subset\tau$ on the group G such that $\sigma|H=\sigma_H$.*

Proof Let $\mathcal{B}_\tau$ and $\mathcal{B}_{\sigma H}$ be bases of unit of (G,τ) and (H,σ_H) respectively.

Put $\mathcal{B}_\sigma=\{U_1U_2 : U_1\in\mathcal{B}_\tau, U_2\in\mathcal{B}_{\sigma H}\}$. Verify that the family $\mathcal{B}_\sigma$ satisfies the Pontrjagin conditions.

2. It is satisfied since $(U_1\cap V_1)(U_2\cap V_2)\subset U_1U_2\cap V_1V_2$.

3. Select $V_2\in\mathcal{B}_{\sigma H}$ and $V_1\in\mathcal{B}_\tau$ such that $V_2^2\subset U_2$, $V_1^2\subset U_1$. Then $(V_1V_2)^2\subset U_1U_2$.

4. Let $y\in U_1U_2$. Then there exist points $y_1\in U_1$ and $y_2\in U_2$ such that $y=y_1y_2$. Therefore there exist a neighborhoods $V_1\in\mathcal{B}_\tau$ and $V_2\in\mathcal{B}_{\sigma H}$ such that $y_iV_i\subset U_i$. Then $yV_1V_2\subset U_1U_2$.

5. It is satisfied since G is abelian.

6. $(U_1^{-1}U_2^{-1})^{-1}\subset U_1U_2$.

1. Since all others Pontrjagin conditions are satisfied then it suffice to show that $\bigcap\mathcal{B}_\sigma=\{e\}$. Let $x\in G$ and $x\neq e$. If $x\in H$ then there exists $U_2\in\mathcal{B}_{\sigma H}$ such that $U_2^2\not\ni x$ and $U_1\in\mathcal{B}_\sigma$ such that $U_1\cap H\subset U_2$. Then $U_1U_2\cap\{x\}=U_1U_2\cap\{x\}\cap H\subset U_2^2\cap\{x\}=\varnothing$. If $x\notin H$ then $(G\backslash xH)H\not\ni x$.

Therefore (G,σ) is a topological group. Since $U_1U_2\cap H=(U_1\cap H)U_2$ then $\sigma|H=\sigma_H$. □

12. Proposition. *A closed subgroup of an H-closed abelian group is H-closed.*

Proof. Let H be a closed subgroup of an H-closed abelian group (G,τ). Then G and H are Rajkov complete. Let $\sigma_H\subset\tau|H$ be a group topology on the group H. Lemma 11 implies that there exists a group topology σ on the group G such that $\sigma|H=\sigma_H$. Let $(\hat G,\hat\sigma)$ be the Rajkov completion of the group (G,σ). Then a closure $\overline{H}^{\hat\sigma}$ of the group H in the group $(\hat G,\hat\sigma)$ is a Rajkov completion of the group (H,σ_H). Let $x\in\overline{H}^{\hat\sigma}$. Theorem 5 implies that there exists $n>0$ such that $x^n\in G$. Since $\overline{H}^{\hat\sigma}\cap G=H$ then $x^n\in H$. Therefore Theorem 5 implies that H is H-closed. □

13. Proposition. *Let G be a H-closed abelian topological group. Then $K=\bigcap_{n\in\mathbb{N}}\overline{nG}$ is compact and for each neighborhood U of zero in G there exists a natural n with $\overline{nG}\subset KU$.*

Proof. Let Φ be a filter on G with a base $\{\overline{nG}:n\in\mathbb{N}\}$, and Ψ be an arbitrary ultrafilter on G with $\Psi\supset\Phi$. Let U be a closed neighborhood of the unit in G. Lemma 2 implies that there exists a number n such that the set $\overline{nG}$ is U-bounded. Since $\overline{nG}\in\Phi$ and Ψ is an ultrafilter, there exists $g\in G$ with $gU\in\Psi$. Hence Ψ is a Cauchy filter on G. By the completeness of G, Ψ is convergent. Therefore each ultrafilter Ψ on G with $\Psi\supset\Phi$ converges. In particular each ultrafilter on K is convergent, and since K is closed, K is compact.

To show that there exists a number n with $\overline{nG}\subset KU$ it suffices to prove that $KU\in\Phi$. Assume that $KU\notin\Phi$. Then there exists an ultrafilter $\Psi\supset\Phi$ with $G\backslash KU\in\Psi$. As we have proved, Ψ is convergent. Clearly $\lim\Psi\in K$. Therefore $KU\in\Psi$ which is a contradiction. Hence $KU\in\Phi$, and this completes the proof. □

14. Corollary. *A divisible abelian H-closed topological group is compact.* □

15. Proposition. *Every H-closed abelian topological group is a union of compact groups.*

Proof. Let G be such a group. It suffice to show that every element $x \in G$ is contained in a compact subgroup. Let X be the smallest closed subgroup of G containing the element x. Then $X = \bigcup_{k=0}^{n}(kx + \overline{nX})$ for every natural n. Let U be an arbitrary neighborhood of the zero. By Lemma 15 there exists a natural number n such that nG is U-bounded. Then X is also U-bounded. Hence X is a precompact group. Since X is Rajkov complete then X is compact. $\square$

16. Conjecture. An abelian topological group G is H-closed if and only if G is Rajkov complete and nG is precompact for some natural n.

17. Proposition. *The Conjecture 16 is true provided the group (G, τ) satisfies the following two conditions:*

(1) There exists a σ-compact subgroup L of G such that G/L is periodic.

(2) There exists a group topology $\tau' \subset \tau$ such that the Rajkov completion $\hat{G}$ of the group (G, τ') is Baire.

Proof. Let G be such a group and $L = \bigcup_{k\in\mathbb{N}} L_k$ be a union of compact subsets L_k. Put $G(n,k) = \{x \in \hat{G} : nx \in L_k\}$ for every natural n and k. Then every set $G(n,k)$ is closed. By Theorem 5 holds $\hat{G} = \bigcup_{n,k\in\mathbb{N}} G(n,k)$. Since $\hat{G}$ is Baire then there exist natural numbers n and k such that $\operatorname{int} G(n,k) \neq \varnothing$. Then $F = G(n,k) - G(n,k)$ is a neighborhood of the zero. By Corollary 6 the group $\hat{G}$ is H-closed. Put $K = \bigcap_{n\in\mathbb{N}} \overline{n\hat{G}}$. By Proposition 13 there exists a natural m such that $m\hat{G} \subset F + K$. Then $mnG \subset mn\hat{G} \subset L_k - L_k + K$ and hence the group mnG is precompact. $\square$

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